The Noonan-Zeilberger Conjecture

Scott Garrabrant
Joint Work with Igor Pak

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Basic Definitions

$A_n(\pi_1, \ldots, \pi_k; m_1, \ldots, m_k)$ is the number of $n \times n$ permutation matrices with exactly m_k instances of π_k as a (not necessarily consecutive) sub matrix.

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$\{A_n\}$ is called **P-recursive** if it satisfies a nontrivial recurrence relation of the form:

$$A_n p_0(n) + A_{n-1} p_1(n) + \ldots + A_{n-\ell} p_\ell(n) = 0$$

where each p_i is a polynomial.
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e.g. $A_n(\pi; 0) = \frac{1}{n+1}\binom{2n}{n}$ for any 3×3 permutation matrix π

$$\frac{1}{n+1}\binom{2n}{n}(n^2) + \frac{1}{n}\binom{2n-2}{n-1}(2 - 2n - 4n^2) = 0$$

The Noonan-Zeilberger Conjecture

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Theorem (G., Pak 2015+)

The Noonan-Zeilberger conjecture is false.
There exists a list of patterns $\pi_1, \ldots, \pi_k$ *of dimension at most* 248×248 *such that* $A_n(\pi_1, \ldots, \pi_k; 0, \ldots, 0)$ *is not P-recursive.*

The Noonan-Zeilberger Conjecture is False

Key Ideas:

- Do everything mod 2.
- Use an involution to require that every instance of pattern B is contained in an instance of some larger pattern A.
- Simulate an automaton using pattern avoidance, inclusion exclusion, and this new involution tool.
- Show this automaton is not always well behaved mod 2.
- Show P-recursive sequences are well behaved mod 2.

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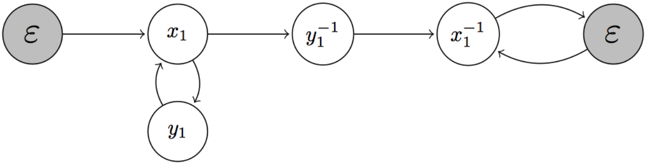
Lemma (G., Pak 2015+)

Let $\{a_n\}$ be a P-recursive integer sequence. Consider an infinite binary word $\mathbf{w} = w_1 w_2 \ldots$ defined by $w_n = a_n \bmod 2$. Then, there exists a finite binary word which is not a subword of w.

Two Stack Automata

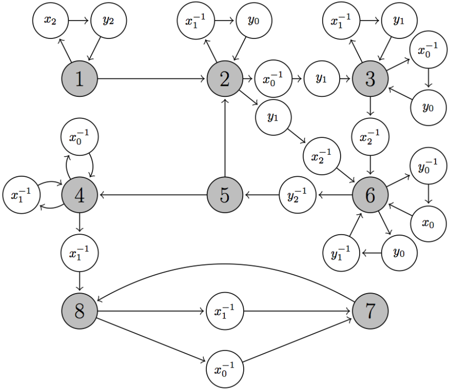


- Finite directed graph Γ, vertices labeled x_i, y_i, x_i^{-1}, y_i^{-1}, ε.
- Traverse paths, keeping two stacks, w_X and w_Y.
- x_i: push x_i onto w_X.
- x_i^{-1}: pop x_i off of w_X.
- w_Y is similar.
- Valid paths end with both stacks empty.
- $G(\Gamma, n)$ is number of valid paths of length n from v_1 to v_2.

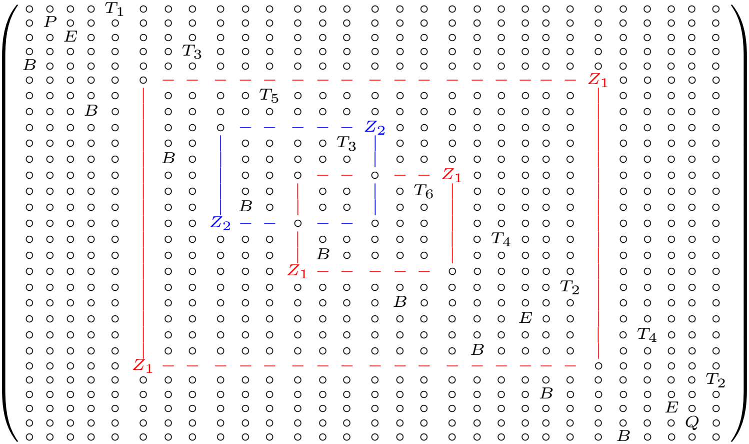
Two Stack Automata mod 2

Lemma (G., Pak 2015+)

There exists a two stack automaton Γ*, such that the word,* $w_1 w_2 \ldots$ *given by* $w_n = G(\Gamma, n)$ *is an infinite binary word which contains every finite binary word as a subword.*



Simulating Two Stack Automata



Each entry is actually a 31×31 matrix

We want to ensure that every B is the $\star$ in a matrix like:

$$\begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_j & \circ & \circ \\ B & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & Z_p \\ \circ & \circ & \circ & \circ & \circ & \circ & T_k & \circ \\ \circ & \star & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & B & \circ & \circ & \circ \\ \circ & \circ & \circ & Z_p & \circ & \circ & \circ & \circ \end{pmatrix}$$

We make B and B' mostly interchangeable but forbid matrices like:

$$\begin{pmatrix} \circ & \circ & T_i & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & T_j & \circ & \circ \\ L & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ & \circ & \circ & Z_p \\ \circ & \circ & \circ & \circ & \circ & \circ & T_k & \circ \\ \circ & B' & \circ & \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & R & \circ & \circ & \circ \\ \circ & \circ & \circ & Z_p & \circ & \circ & \circ & \circ \end{pmatrix}, \quad L, R \in \{B, B'\}$$

The desired matrices are the fixed points of an involution swapping B and B' whenever possible.

Putting it all Together

Lemma (G., Pak 2015+)

Let Γ *be a two stack automaton. There exist integers* c *and* d *and two lists* $\pi_1, \ldots, \pi_k$ *and* $\pi'_1, \ldots, \pi'_{k'}$*, such that*

$$A_{cn+d}(\pi_1, \ldots, \pi_k; 0, \ldots, 0) - A_{cn+d}(\pi'_1, \ldots, \pi'_{k'}; 0, \ldots, 0) = G(\Gamma, n)$$

modulo 2.

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Future Work

- How many patterns does it take to construct a counter example to the Noonan-Zeilberger conjecture?
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  We think no, but are still working on some details.

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  Somewhere between 1 and 10^{488}.
- Does $\{A_n(\pi_1, \ldots, \pi_k; m_1, \ldots, m_k)\}$ always have a generating function satisfying an algebraic differential equation?
  We think no, but are still working on some details.
- Is it decidable whether or not $A_n(\pi_1, \ldots, \pi_k; m_1, \ldots, m_k) = A_n(\pi'_1, \ldots, \pi'_k; m'_1, \ldots, m'_k)$?
  It is not decidable if they are equal mod 2.

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