Counting with Irrational Tiles

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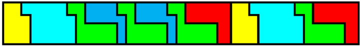
- *Irrational Tile* – an axis-parallel polygon
- T denotes a set of tiles.
- $C_T(\Gamma)$ = the number of tilings of a region Γ by T.
- $R_\varepsilon(n)$ denotes the $[1 \times (n+\varepsilon)]$ rectangle.



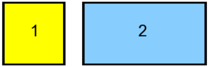
Main Question

Which functions "count tilings of a rectangle?"

I.e. for which functions f does $f(n) = C_T(R_\varepsilon(n))$ for some tile set T of height 1 irrational tiles and $\varepsilon \in \mathbb{R}^{\geq 0}$?

Example - Fibonacci Numbers

- $f(n) = F_n$, the n^{th} Fibonacci number.



$\varepsilon = 0$

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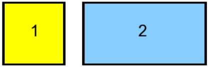

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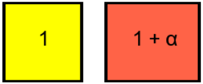
Example: $n = 21, \Gamma = [1 \times 21]$

Example - n

- $f(n) = n$



$\varepsilon = \alpha$

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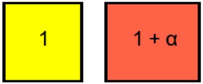

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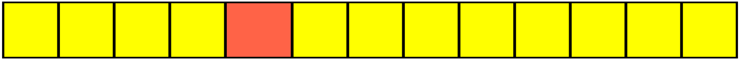
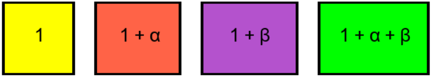
Example: $n = 13, \Gamma = [1 \times (13 + \alpha)]$



Example - n^2

- $f(n) = n^2$



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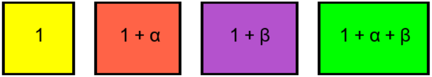

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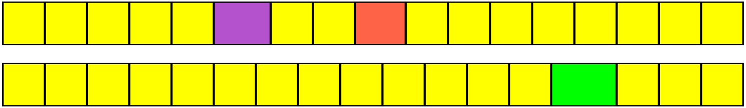
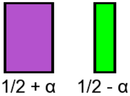
Example: $n = 17, \Gamma = [1 \times (17 + \alpha + \beta)]$



Example - $\binom{2n}{n}$

- $f(n) = \binom{2n}{n}$



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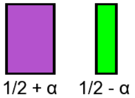
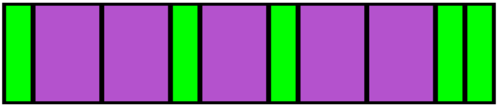


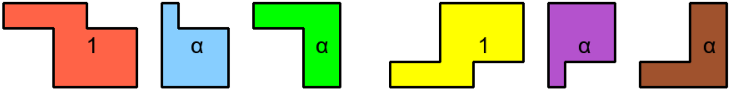
$\varepsilon = 0$

Example: $n = 5, \Gamma = [1 \times 5]$



Example - 2

- $f(n) = 2$



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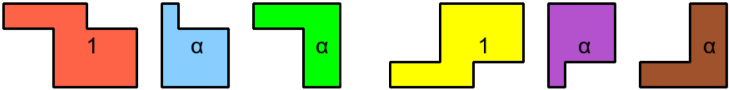

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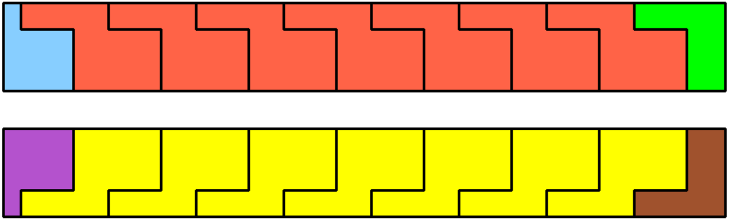
Example: $n = 7$, two tilings of $\Gamma = [1 \times (7 + 2\alpha)]$



- Let $G \in \mathbb{Z}[[x_1, \ldots, x_k]]$ be a rational generating function, corresponding to the rational function P/Q for some polynomials $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$.
- The diagonal of G is the function $d(n) := [x_1^n, \ldots, x_k^n]\, G$.

Diagonals of Rational Generating Functions

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- The diagonal of G is the function $d(n) := [x_1^n, \ldots, x_k^n]\, G$.
- Example: $\binom{2n}{n}$ is the diagonal of $\frac{1}{1-x_1-x_2}$

Let the class $\mathcal{R}_k$ of $\mathbb{N}$-rational generating functions be the smallest set of rational generating functions which satisfies:

1. $0, x_1, \ldots, x_k \in \mathcal{R}_k$
2. $F, G \in \mathcal{R}_k \implies F + G, F \cdot G \in \mathcal{R}_k$
3. $F \in \mathcal{R}_k, [1]F = 0 \implies \frac{1}{1-F} \in \mathcal{R}_k$

Generating functions in $\mathcal{R}_k$ have all non-negative coefficients, but not all non-netative coefficient rational generating functions are in $\mathcal{R}_k$.

- Let $\mathcal{F}$ be the set of all tile counting functions
- Let $\mathcal{D}$ be the set of all diagonals of functions in $\mathcal{R}_k$.

Theorem (G., Pak 2014+)

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Theorem (G., Pak 2014+)

$$\mathcal{F} = \mathcal{D}$$

Corollary (G., Pak 2014+)

$\mathcal{F}$ is closed under addition and multiplication.

Binomial Coefficients

Notation:

1. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ when $n \geq k \geq 0$
2. $\binom{-1}{0} = 1$
3. $\binom{n}{k} = 0$ otherwise.

This comes from defining $\binom{n}{k}$ as the number of ways of dividing k indistinguishable objects into $n - k + 1$ indistinguishable groups.

Binomial Multisums

Let $\mathcal{B}$ denote the set of all functions $f : \mathbb{N} \to \mathbb{N}$ which can be expressed as

$$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

where each α_i and β_i is an integer coefficient affine function of v and n. I.e. $\alpha_i(v,n) = z_1 v_1 + \ldots + z_d v_d + z_{d+1} n + z_{d+2}, z_i \in \mathbb{Z}$

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$$\mathcal{F} = \mathcal{D} = \mathcal{B}$$

Examples

$$F_n = \binom{n}{0} + \binom{n-1}{1} + \ldots + \binom{\lceil n/2 \rceil}{\lfloor n/2 \rfloor} = \sum_{v \in \mathbb{Z}} \binom{n-v}{v} \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n$$

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$$f(n) = \sum_{j,k \in \mathbb{Z}} \binom{n-k}{n-k} \binom{7n-7k}{j} \binom{4k}{k} \binom{3k}{k} =$$

$$\sum_{k=0}^{n} 128^{n-k} \binom{4k}{k, k, 2k} \sim \; {}_2F_1 \left(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{2} \right) 128^n = \frac{128^n \sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)}$$

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$$\phi(n) = \begin{cases} \binom{n}{\frac{n}{4}, \frac{n}{4}, \frac{n}{4}, \frac{n}{4}} & \text{if } n = 0 \mod 4, \\ 4\binom{n-1}{\frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}} & \text{if } n = 1 \mod 4, \\ 64\binom{n-2}{\frac{n-2}{4}, \frac{n-2}{4}, \frac{n-2}{4}, \frac{n-2}{4}} & \text{if } n = 2 \mod 4, \\ 256\binom{n-3}{\frac{n-3}{4}, \frac{n-3}{4}, \frac{n-3}{4}, \frac{n-3}{4}} & \text{if } n = 3 \mod 4. \end{cases} \sim \frac{32}{\pi} \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

Proof Sketch ($\mathcal{D} \subseteq \mathcal{F}$)

- Given an N-rational generating function G in k variables, construct a network with k colored edges.
- The number of paths through the network using exactly n_i edges of color i for each $1 \leq i \leq k$ is equal to $\left[x_1^{n_1}, \ldots, x_k^{n_k}\right] G$
- Constructed recursively using the definition of N-rational.
- Convert this network to a tile set using edges as tiles, vertices as boundaries between tiles, and irrational areas to enforce that there are n edges of each color.

Proof Sketch ($\mathcal{B} \subseteq \mathcal{D}$)

- Prove that there exists a linear bound on vectors v which contribute to the sum. I.e.

$$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)} = \sum_{v \in \mathbb{Z}^d, \frac{|v_i|}{n} \le M} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

  for some $M \in \mathbb{N}$.
- Use this M to explicitly construct a generating function representing sum over all choices of $-Mn \le v_i \le Mn$ of a product of binomial coefficients.
- This construction is a large product of terms which are each clearly $\mathbb{N}$-rational.

Proof Sketch ($\mathcal{F} \subseteq \mathcal{B}$)

- Convert the tiling question to a question of cycles in a weighted directed graph of a given weight.
- Decompose a cycle into a unique list of "irreducible cycles."
- $$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)}$$
- The v vector roughly represents the multiplicity of the irreducible cycles in the decomposition
- The $\binom{\alpha_i(n,v)}{\beta_i(n,v)}$ represent choices that must be made in piecing these irreducible cycles together into one cycle.

Useful Corollary

Theorem (G., Pak 2014+)

If $f(n)$ counts tilings of a rectangle, there exists $m \geq 1$, such that $f_k(n) = f(nm + k)$, $0 \leq k \leq m - 1$ satisfies either

$$f_k = e^{\Theta(n)} \quad \text{or} \quad f_k(n) = p(n),$$

for some polynomial p, for all sufficiently large n.

Proof uses integer points in polytopes and classical results about Ehrhart polynomials.

Non-examples

Corollary (G., Pak 2014+)

The following functions do NOT count tilings of a rectangle:

- $\lfloor \log_2(n) \rfloor = o(n)$
- $\lfloor \sqrt{n} \rfloor = o(n)$
- $\sigma_0(n) =$ *the number of divisors of* n, $\sigma(n) = o(n)$
- $p(n) =$ *the number of integer partitions of* n, $p(n) = e^{\Theta(\sqrt{n})}$
- $g(n) =$ *the number of connected labeled graphs on* $n+1$ *vertices*, $g(n) = e^{\Theta(n^2)}$

This follows immediately from asymptotic arguments.

Mysterious Example: Catalan Numbers

Conjecture (G., Pak 2014+)

The Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$ *do not count tilings of a rectangle.*

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Claim

Asymptotic methods will not prove the conjecture.

Theorem (G., Pak 2014+)

For every $\varepsilon > 0$*, there exists a function* f *which counts tilings of a rectangle, such that* $\frac{f(n)}{C_n} \to \lambda$*, where* $1 - \varepsilon < \lambda < 1 + \varepsilon$*.*

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Sketch of Proof:

$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$.

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$\phi(n-i) \sim \frac{1}{4^i}\frac{32}{\pi}\frac{4^n}{n^{3/2}\sqrt{\pi}}$ for any fixed i.

Approximate $\frac{\pi}{32}$ as a sum of numbers of the form $\frac{1}{4^i}$. □

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$$f(n) = \binom{2n}{n} + (m-1)\binom{2n}{n+1} = \binom{2n}{n} - \binom{2n}{n+1} = C_n \mod m$$

Thank You!